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Theoretical Perspectives on Representations of Plesken Lie Algebras

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An prominent field of study related to Lie theory, representations of Lie groups and Lie algebras, has its origins in the work of Sophus Lie, who examined certain transformation groups that are now known as Lie groups. Both Lie groups and Lie algebras are now fundamental to many areas of theoretical physics and mathematics. Compared to the Liegroup, the Lie algebra is more readily available since it is a linear entity. Wilhelm Killing (1847–1923) argued that categorising all (finite dimensional real) Lie algebras should come before classifying all group actions. It became evident from the slow development of Lie, Friedrich Engel (1861–1941), and Killing's theories that figuring out all basic Lie algebras was essential. Unless otherwise indicated, a Lie algebra is a finite dimensions over a field with characteristic zero in this article.

A vector space with a basis made up of group elements and additional structure using the product operation on G is called a group algebra of a finite group G. Group algebras may be thought of as the foundation for representation theory of finite groups. A homomorphism ρ: G GL(V) of G to the group of automorphisms of V is a representation of a finite group G on a finite dimensional complex vector space V. A vector subspace W of a representation V that is invariant under G is a subrepresentation of that representation. For any g G and w W, that is ρ(g)(w) W. If there isn't a valid invariant subspace W of V, then ρ is irreducible. Through the bracket (commutator) operation, the group algebra takes on the structure of a Lie algebra.

Plesken Lie algebra is the name given to the creation of a Lie algebra of a finite group by W. Plseken and Arjeh M. Cohen. They ran into the issue of which groups the construction would provide a simple Lie algebra for. By explicitly identifying the groups for which the Plesken Lie algebra is simple or semisimple, Arjeh M. Cohen and D. E. Taylor conducted a thorough investigation of the structure of Plesken Lie algebras in [1]. The representations of Plesken Lie algebras, Plesken Lie algebra modules, and the representations of Plesken Lie algebras derived from the group representations are all covered in this work. Additionally, we identify the irreducible representations of Plesken Lie algebras.

1. PRELIMINARIES

We will first go through preliminaries of Lie algebras, group algebras and PleskenLie algebras.Also we describes some structural properties of group algebra.Fur-ther we will introduce the notion of representation of Plesken Lie algebras. We alsodiscusssomeirreduciblerepresentationsofPleskenLiealgebrasofsomegroups.

ALie algebra$L$overanarbitraryfieldFisavectorspaceoverFendowedwithanoperationcalledLiebracketsatisfying thefollowingproperties:

(1) Bilinearity:For$x,y,z\in L,a,b\in$F

$[ax+by,z]=a[x,z]+b[y,z]$

$[x,ay+bz]=a[x,y]+b[x,z]$

(2) $[x,x]=0$forall$x\in L$

(3) Jacobiidentity:

$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$forall$x,y,z\in L$

A vector subspace $K \subseteq L$ is a *Lie subalgebra*of L if $[x, y] \in K$ for all $x, y \in K$, andasubspace I of a Lie algebra L is called an *ideal* if $[x, y]\in I$ for all $x \in L$, $y \in I$.A *Lie algebra homomorphism* from L to $L'$is a linear map $\varphi : L \to L'$such that$\varphi([x,y])=[\varphi(x),\varphi(y)]$forall$x,y \in L$.

AgroupalgebraFGofagroup$G$overFisavectorspacewithgroupelemenstsasbasis.That is,

$$FG=\{\sum_{i=1}^{n} a_ig_i : a_i\in F \text{ for all } i\}$$

Herethemultiplicationisdefinedby

$$\left(\sum_{i=1}^{n} a_ig_i\right)\left(\sum_{j=1}^{n} b_jg_j\right)=\sum_{i,j=1}^{n} a_ib_j(g_ig_j),a_i,b_j\in F$$

AlinearmapbetweentwogroupalgebrasFGandFHwhichpreservesthealgebramultiplicationisa*homomorphism*ofgroupalgebras.

Definition1.*(cf.*[1]*)PleskenLiealgebra*L(G)*ofagroupGover*F*isthelinearspanof elements*$\hat{g}=g-g^{-1}\in$FG*togetherwiththeLiebracket*

$$[\hat{g},\hat{h}]=\hat{g}\hat{h}-\hat{h}\hat{g}$$

Example1.*ConsiderthesymmetricgroupS*$_3$*,then*

$L(S_3)=span\{\sigma-\sigma^{-1}:\sigma\in S_3\}$
$=\{a_1((1)-(1))+a_2((12)-(12))+a_3((13)-(13))++a_4((23)-(23))$
$+a_5((123)-(132))+a_6((132)-(123)):a_i\in C\}$
$=\{a((123)-(132)):a\in C\}$

isaonedimensional Plesken Liealgebraover C*withLiebracket*

$[a(\widehat{123}), b(\widehat{123})]=0$

2. GROUPALGEBRAHOMOMORPHISMSANDPLESKENLIEALGEBRA HOMOMORPHISMS

Herewegothroughsomestructuralpropertiesofgroupalgebrasand PleskenLie algebras such as group algebra homomorphisms, group algebra representationsandPleskenLiealgebrahomomorphisms.

Lemma 1. *Let $f: G \to H$ be a group homomorphism. Then $\bar{f}: \mathbb{F}G \to \mathbb{F}H$ defined by*

$$\bar{f}\left(\sum_{i=1}^{n} a_i g_i\right) = \sum_{i=1}^{n} a_i f(g_i)$$

is a homomorphism between group algebras.

Proof. Let $G = \{g_1, g_2, \ldots, g_n\}$, then $\mathbb{F}G = \{\sum_{i=1}^{n} a_i g_i : a_i \in \mathbb{F} \text{ for all } i\}$. Clearly $\bar{f}$ is linear and for $\alpha = \sum_{i=1}^{n} a_i g_i$, $\beta = \sum_{j=1}^{n} b_j g_j \in \mathbb{F}G$,

$$\bar{f}\left(\sum_{i=1}^{n} a_i g_i \sum_{j=1}^{n} b_j g_j\right) = \bar{f}\left(\sum_{i,j=1}^{n} a_i b_j g_i g_j\right) = \sum_{i,j=1}^{n} a_i b_j f(g_i) f(g_j) = \sum_{i=1}^{n} a_i f(g_i) \sum_{j=1}^{n} b_j f(g_j)$$

hence, $\bar{f}$ is a homomorphism between group algebras. Q

However, it is not necessary that any group algebra homomorphisms induced by a group homomorphisms as seen in the following example.

Example 2. *Consider the group $G = \{e, \sigma\}$ where $\sigma^2 = e$. Define $f: \mathbb{C}G \to \mathbb{C}G$ by $f(e) = e$ and $f(\sigma) = -\sigma$ with f is linear. Then f is a group algebra homomorphism. But f is not induced from any group homomorphism.*

A representation of a group algebra $\mathbb{F}G$ is a linear map $\varphi : \mathbb{F}G \to \mathrm{gl}(V)$ (where $\mathrm{gl}(V)$ is the set of all endomorphisms on V) such that φ is an algebra homomor-phism. A representation of a group G gives rise to a representation of the group algebra $\mathbb{F}G$ as follows.

Definition 2. *Given a representation $\rho: G \to GL(V)$ of a group G. Then $\varphi: \mathbb{F}G \to \mathrm{gl}(V)$ defined by*

$$\varphi\left(\sum_{i=1}^{n} a_i g_i\right) = \sum_{i=1}^{n} a_i \rho(g_i)$$

is a representation of $\mathbb{F}G$.

A *subrepresentation* of a representation $\varphi: \mathbb{F}G \to \mathrm{gl}(V)$ is a map $\varphi' : \mathbb{F}G \to \mathrm{gl}(W)$ where W is a vector subspace V which is invariant under $\mathbb{F}G$. A representation φ is *irreducible* if there is no proper $\mathbb{F}G$-invariant subspace W of V. That is, there is no proper invariant subspace W such that $\varphi(x)(w) \in W$ for all $w \in W$ and $x \in \mathbb{F}G$.

Theorem 1. *If $\rho: G \to GL(V)$ is an irreducible representation of G, then the representation $\varphi: \mathbb{F}G \to \mathrm{gl}(V)$*

definedby

$$\varphi\left(\sum_{i=1}^{n} a_i g_i\right) = \sum_{i=1}^{n} a_i \rho(g_i)$$

isalsoirreducible.

Proof. Since $\rho : G \to GL(V)$ is an irreducible representation, V has no proper G-invariant subspace. Let $\varphi : \mathbb{F}G \to \mathrm{gl}(V)$ be the corresponding representation of $\mathbb{F}G$. Also let W be an invariant subspace of V under the action φ. That is $\varphi(x)(w) \in W$ for all $w \in W$ and $x \in \mathbb{F}G$.

$$\varphi(x)(w) = \varphi\left(\sum_i^n a_i g_i\right)(w) = \sum_i^n a_i \rho(g_i)(w)$$

In particular, $\rho(g_i)(w) \in W$ for all $w \in W$ and $g_i \in G$. That is W is an invariant subspace of V under the action ρ, which is a contradiction since ρ is irreducible. Hence there is no invariant subspace W of V under the action φ. Q

The following example shows that the group algebra representations of the irreducible representations of the symmetric group S_3 are also irreducible.

Example 3. *Consider the symmetric group S_3 and its irreducible representations ρ_1, ρ_2 and ρ_3 where ρ_1 is the trivial representation with degree 1, ρ_2 is the alternating representation with degree 1 and ρ_3 is the standard representation with degree 2. Also,*

$\rho_1 : \sigma \mapsto [1]$ *for all* $\sigma \in S_3$

ρ_2: *even permutations maps to [1] and odd permutations maps to [-1]*

$$\rho_3 : (123) \mapsto \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad (12) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then from Theorem 2, the group algebra representations are:

$\varphi_1 : \mathbb{R}S_3 \to \mathrm{gl}(\mathbb{R})$ *given by*

$$\varphi_1(a_1(1)+a_2(12)+a_3(23)+a_4(13)+a_5(123)+a_6(132)) = (a_1+a_2+a_3+a_4+a_5+a_6)1$$

$\varphi_1 : \mathbb{R}S_3 \to \mathrm{gl}(\mathbb{R})$ *given by*

$$\varphi_2(a_1(1)+a_2(12)+a_3(23)+a_4(13)+a_5(123)+a_6(132)) = (a_1+a_5+a_6)[1]+(a_2+a_3+a_4)[-1]$$

$\varphi_3 : \mathbb{R}S_3 \to \mathrm{gl}(\mathbb{R}^2)$ *given by*

$$\varphi_2(a_1(1)+a_2(12)+a_3(23)+a_4(13)+a_5(123)+a_6(132)) = a_1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + a_3\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} + a_4 \cdots + a_5 \cdots$$

3
2 $+a_6$

2

—
1
3
—
√
2
3
2
2

2

−1 ——

− −1

Since $\varphi_1$amd $\varphi_2$are of degree 1, they are irreducible. To prove $\varphi_3$is irreducible,let W be a proper invariant subspace of $\mathbb{R}^2$. *Then W= span*$\{(\alpha, \beta)\}$ *for some*$(\alpha,\beta)\in\mathbb{R}^2$.*SinceWisinvariant,* $\varphi_3(x)(w)\in W$*forall*$x\in\mathbb{R}S_3$*and*$w\in W$.

That is, $\varphi(\sum_{i=1}^{n} a_i\sigma_i)(w)=k(\alpha,\beta)$ for some $k\in\mathbb{R}$. From computations we obtained,

$$\alpha\left(a_1+a_2-\frac{a_3}{2}-\frac{a_5}{2}-\frac{a_6}{2}\right)+\frac{\sqrt{3}\beta}{2}(a_3+a_4-a_5-a_6)=k\alpha$$

$$\frac{\sqrt{3}\alpha}{2}(a_4-a_3+a_5-a_6)+\beta\left(a_1-a_2+\frac{a_3}{2}+\frac{a_4}{2}-\frac{a_5}{2}-\frac{a_6}{2}\right)=k\beta$$

This is true for every $x\in\mathbb{R}S_3$ *and* $w\in W$. *Thus take* $x=\sqrt{2}(23)$, *then it can be seen that* $\alpha=\beta=0$. *Thus* $W=0$ *and hence* φ_3 *is irreducible.*

Definition 3. *Let G be a finite group and* F *be a field. The group algebra* F*G over* F *with the Lie bracket* $[,]:\mathbb{F}G\times\mathbb{F}G\to\mathbb{F}G$ *defined by*

$$[\alpha,\beta]=\alpha\beta-\beta\alpha \text{ where } \alpha=\sum_i a_ig_i,\ \beta=\sum_i b_ig_i \text{ in } \mathbb{F}G$$

is a Lie algebra, denote it by $L_{\mathbb{F}G}$ *and is called the Group Lie algebra.*

A linear map between two group Lie algebras $L_{\mathbb{F}G}$ and $L_{\mathbb{F}H}$ which preserves the Lie bracket is a *homomorphism* of group Lie algebras.

The following lemmas are immediately follow from the definitions.

Lemma 2. *Let* $f:G\to H$ *be a group homomorphism. Then* $\bar{f}:L_{\mathbb{F}G}\to L_{\mathbb{F}H}$ *defined by*

$$\bar{f}\left(\sum_{i=1}^{n} a_ig_i\right)=\sum_{i=1}^{n} a_if(g_i)$$

is a homomorphism between group Lie algebras.

Proof. Let $G=\{g_1,g_2,\dots,g_n\}$, then $L_{\mathbb{F}G}=\{\sum_{i=1}^{n} a_ig_i : a_i\in\mathbb{F} \text{ for all } i\}$. Clearly $\bar{f}$ is linear and for $\alpha=\sum_{i=1}^{n} a_ig_i$, $\beta=\sum_{j=1}^{n} b_jg_j\in L_{\mathbb{F}G}$,

$$\bar{f}([\alpha,\beta])=\bar{f}(\alpha\beta-\beta\alpha)$$

$$=\bar{f}\Big(\sum_{i,j=1}^{n} a_i b_j g_i g_j-\sum_{j,i=1}^{n} b_j a_i g_j g_i\Big)$$

$$=\sum_{i,j=1}^{n} a_i b_j f(g_i) f(g_j)-\sum_{j,i=1}^{n} b_j a_i f(g_j) f(g_i)$$

$$=\sum_{i=1}^{n} a_i f(g_i)\sum_{j=1}^{n} b_j f(g_j)-\sum_{j=1}^{n} b_j f(g_j)\sum_{i=1}^{n} a_i f(g_i)$$

$$=[\bar{f}(\alpha),\bar{f}(\beta)]$$

hence, $\bar{f}$ is a homomorphism between Lie algebras of group algebras. Q

Clearly the Plesken Lie algebra L(G) is a proper subset of L_{FG} and moreover, it is a Lie subalgebra of L_{FG}.

Example4. *Consider the Plesken Lie algebras* $\mathcal{L}(S_3) = \{\alpha((1\ 2\ 3) - (1\ 3\ 2)) : \alpha \in \mathbb{C}\}$ *of* S_3 *over* $\mathbb{C}$ *and* $\mathcal{L}(D_4) = \{\alpha(a - a^3) : \alpha \in \mathbb{C}\}$ *of* $D_4 = \langle a, b : a^4 = b^2 = e, aba = b^{-1} \rangle$ *over* $\mathbb{C}$. *Then* $\hat{f}: \mathcal{L}(S_3) \to \mathcal{L}(D_4)$ *defined by*

$$\hat{f}(\alpha((123)-(132)))=\alpha(a-a^3)$$

is a Plesken Lie algebra homomorphism.

Lemma3. *Let* $f: G \to H$ *be a group homomorphism. Then* $\hat{f}: \mathcal{L}(G) \to \mathcal{L}(H)$ *defined by*

$$\hat{f}\left(\sum_{i=1}^{n} a_i \hat{g}_i\right) = \sum_{i=1}^{n} a_i \widehat{f(g_i)}$$

is a Plesken Lie algebra homomorphism. Further, $\hat{f}$ *is actually the restriction of the homomorphism of the group Lie algebras* $\bar{f}: L_{FG} \to L_{FH}$ *(defined in Lemma 3) to* $\mathcal{L}(G)$. *That is,*

$$\hat{f} = \bar{f}|_{\mathcal{L}(G)}$$

Proof. The proof is similar as in *Lemma 3*. ∎

3. PLESKEN LIE ALGEBRA REPRESENTATIONS

Next we proceed to define representation of a Plesken Lie algebra and discuss some of its properties. We have already described the representations of group algebras and it is seen that when the representations of group is irreducible so is the representation of group algebras.

Definition4. *A representation of a Plesken Lie algeba* $\mathcal{L}(G)$ *is a linear map* $\varphi: \mathcal{L}(G) \to gl(V)$ *(where V is a vector space over* F*) such that* φ *is a Lie algebra homomorphism.*

The following theorem shows that if we have a representation of a group G, then we can find a representation of the Plesken Lie algebra $\mathcal{L}(G)$.

Lemma4. *If* $\rho: G \to GL(V)$ *is a representation of G on V, then* $\psi: \mathcal{L}(G) \to gl(V)$ *defined by*

$$\psi\left(\sum_{i=1}^{n} a_i \hat{g}_i\right) = \sum_{i=1}^{n} a_i \widehat{\rho(g_i)} \text{ for } \sum_{i=1}^{n} a_i \hat{g}_i \in \mathcal{L}(G)$$

is a representation of the Plesken Lie algebra $\mathcal{L}(G)$.

Proof. Since $\rho: G \to GL(V)$ is a repesentation of G, $\rho(g_i)$ and $\rho(g_i^{-1})$ are auto-morphisms on V. Then

$$\widehat{\rho(g_i)} = (\rho(g_i) - \rho(g_i)^{-1}) = \rho(g_i) - \rho(g_i^{-1}).$$

is an endomorphism on V. Thus ψ is defined.
Clearly ψ is linear. By *Lemma* 4, ψ is a homomorphism from $\mathcal{L}(G)$ to $gl(V)$ and hence ψ is a representation of $\mathcal{L}(G)$. ∎

A Plesken Lie algebra representation $\psi: \mathcal{L}(G) \to gl(V)$ is *irreducible* if there is no proper $\mathcal{L}(G)$-invariant subspace W of V.

The following theorem states that the Plesken Lie algebra representation corresponding to a reducible group representation is reducible.

Theorem 2. *If $\rho : G \rightarrow GL(V)$ is a reducible representation of a group G, then the Plesken Lie algebra representation ψ: $L(G) \rightarrow gl(V)$ of $L(G)$ defined by*

$$\psi\left(\sum_{i=1}^{n} a_i \hat{g}_i\right) = \sum_{i=1}^{n} a_i \widehat{\rho(g_i)}$$

is also reducible.

Proof. Since $\rho : G \rightarrow GL(V)$ is a reducible representation, V has a proper invariant subspace. That is, there is a subspace W of V such that $\rho(g)w \in W$ for all $g \in G$ and $w \in W$. We claim that W is also invariant under ψ. For let $\hat{x} = \sum_{i=1}^{n} a_i \hat{g}_i \in L(G)$, $w \in W$,

$$\psi(\hat{x})(w) = \psi\left(\sum_{i=1}^{n} a_i \hat{g}_i\right)(w) = \sum_{i=1}^{n} a_i \widehat{\rho(g_i)}(w) = \sum_{i=1}^{n} a_i(\rho(g_i)(w) - \rho(g_i^{-1})(w))$$

Since $\rho(g_i)(w), \rho(g_i^{-1})(w) \in W$ and W is a subspace, $\widehat{\rho(g_i)}(w) = \rho(g_i)(w) - \rho(g_i^{-1})(w) \in W$.
Thus $\psi(\hat{x})w \in W$ for all $\hat{x} \in L(G)$ and $w \in W$. Q

Very often if ρ is an irreducible representation of a group G, then ψ is also an irreducible representation as seen in the following examples (where ψ is defined as in *Lemma* 5).

Example 5. *Consider the irreducible representations of S_3 as in Example 3. Then corresponding Plesken Lie algebra representations are:*

ψ_1: $L(S_3) \rightarrow gl(\mathbb{R})$ *given by* $\psi_1(a((123) - (132))) = 0$

ψ_2: $L(S_3) \rightarrow gl(\mathbb{R})$ *given by* $\psi_2(a((123) - (132))) = 0$

$$\psi_3: L(S_3) \rightarrow gl(\mathbb{R}^2) \text{ given by } \psi_3(a((123) - (132))) = a\left(\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\right) = a\begin{pmatrix} 0 & -\sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}$$

Clearly, ψ_1 and ψ_2 are irreducible representations since each of which has degree 1. We will prove ψ_3 is irreducible. For W be a proper invariant subspace of $\mathbb{R}^2$. *Then*
$W = span\{(\alpha, \beta)\}$ for some $(\alpha, \beta) \in \mathbb{R}^2$. Since W is invariant, $\psi_3(\hat{x})(w) \in W$ for all $\hat{x} \in L(S_3)$ and $w \in W$.

$$\psi_3(\hat{x})(w) \in W \Rightarrow a\begin{pmatrix} 0 & -\sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in W$$

$$\Rightarrow a\begin{pmatrix} -\sqrt{3}\beta \\ \sqrt{3}\alpha \end{pmatrix}$$

$) \in W$

$$\Rightarrow (-\sqrt{3}a\beta, \sqrt{3}a\alpha) = k(\alpha, \beta) \text{ for some } k \in \mathbb{R}$$

$$\Rightarrow k = \frac{-\sqrt{3}a\beta}{\alpha} \text{ and } k = \frac{\sqrt{3}a\alpha}{\beta}$$

By equating we get, $\sqrt{3}a(\alpha^2+\beta^2)=0$ *which implies* $\alpha=\beta=0$. *That is,* $W=0$, *hence* ψ_3 *is an irreducible representation.*

Example6. *Considerthedihedralgroup* $D_4=<a,b:a^4=b^2=e, aba=b>$ *anditsirreduciblerepresentations* $\rho_1,\rho_2,\rho_3,\rho_4$ *and* ρ_5 *where* ρ_i *'saregivenby*

$$\rho_1: a\mapsto(1), b\mapsto(1)$$
$$\rho_2: a\mapsto(-1), b\mapsto(1)$$
$$\rho_3: a\mapsto(1), b\mapsto(-1)$$
$$\rho_4: a\mapsto(-1), b\mapsto(-1)$$
$$\rho_5: a\mapsto\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, b\mapsto\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

Then correspondingPlesken Liealgebra representations are: $\psi_1 : L(D_4) \to gl(R)$ *given by* $\psi(\alpha(a - a^3)) = 0$ $\psi_2 : L(D_4) \to gl(R)$ *given by* $\psi(\alpha(a - a^3)) = 0$ $\psi_3 : L(D_4) \to gl(R)$ *given by* $\psi(\alpha(a - a^3)) = 0$ $\psi_4: L(D_4)\to gl(R)$ *givenby* $\psi(\alpha(a-a^3))= 0$
$\psi_5: L(D_4)\to gl(R^2)$ *givenby* $\psi(\alpha(a-a^3))=\alpha\begin{pmatrix}0 & 2\\ 2 & 0\end{pmatrix}$

Since ψ_1, ψ_2, ψ_3 *and* ψ_4 *hasdegree1,theyare irreducible.Next wewillprove* ψ_5 *isalso an irreducible representation. For let W be a proper invariant subspace of* R^2. ***Then*** *W=span*(α,β) *forsome* $(\alpha,\beta)\in R^2$. ***Since W*** *isinvariant,* $\psi_5(\hat{x})(w)\in W$ *forall* $\hat{x}\in L(D_4)$ *and* $w\in W$.

$$\psi_5(\hat{x})(w)\in W\Rightarrow\gamma\begin{pmatrix}0 & -2\\ 2 & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\in W$$
$$\Rightarrow\gamma\begin{pmatrix}-2\beta\\ 2\alpha\end{pmatrix}\in W$$
$$\Rightarrow(-2\gamma\beta, 2\gamma\alpha)=k(\alpha,\beta) \text{ forsome } k\in R$$
$$\Rightarrow k=\frac{-2\gamma\beta}{\alpha} \text{ and } k=\frac{2\gamma\alpha}{\beta}$$

Byequatingweget, $2\gamma(\alpha^2+\beta^2)=0$ *whichimplies* $\alpha=\beta=0$. *Thatis,* $W=0$, *hence* ψ_5 *isanirreduciblerepresentation.*

Alsoweobtained thattherepresentations of $L(S_4)$ corresponding totheirre-duciblerepresentationsof S_4 areirreducible.

However,theabovesituationisnotanecessaryandsufficientcondition, becauseitispossibletohave ψ maybe $L(G)$-reducibleevenif ρ is FG-irreducible.

3.1. **PleskenLiealgebramodules.** NextweproceedtodescribesPleskenLiealgebramodulesandobtainsomeinterestingtheoriessuchasSchur'slemma.

Definition5. *AvectorspaceV,endowedwithanoperation* $L(G)\times V\to V$ *isan* $L(G)$*-moduleif*

(1) $(a\hat{x}+b\hat{y})v=a(\hat{x}v)+b(\hat{y}v)$
(2) $\hat{x}(av+bw)=a(\hat{x}v)+b(\hat{x}v)$
(3) $[\hat{x},\hat{y}]=\hat{x}\hat{y}v-\hat{y}\hat{x}v$

forall $\hat{x},\hat{y}\in L(G)$, $v,w\in V$ *and* $a,b\in F$.

Remark3. *Every* FG *isan* $L(G)$*-module.*

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Proof. Suppose V is an FG-module. Then for any $\hat{x} \in$ L(G) and $v \in V$,

$$\hat{x}v = \Big(\sum_{i=1}^{n} \hat{g}_i\Big)v = \sum_{i=1}^{n} a_i(\hat{g}_i v)$$

Since $V$is an FG-module, $gv\in V$ for all $g \in G$ and $v \in V$. Thus $\hat{g}_i v = g_i v - g_i^{-1} v \in V$whichimplies$\hat{x}v\in V$forall$\hat{x}\in$L($G$)and$v\in V$andthissatisfiesalltheaxioms ofanL(G)-module. Thus$V$ isanL(G)-module.
Q

Notethattheconverseoftheremarkneednotbetrue.

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